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Routing problems, multi-commodity fixed-charge network design and variable splitting

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The multi-commodity fixed-charge network design problem (MCFCND) can be defined on a graph $G = (V, A)$ with vertices V and arcs A . Let K be a set of commodities that have to be transported and d_k be amount of commodity k that that needs to be transported. We assume that each commodity has a single origin node and a single destination node denoted $s(k)$ and $e(k)$, respectively. For each node $i \in V$ and commodity $k \in K$ we define

$$d(i, k) = \begin{cases} d_k & \text{if } i = s(k) \\ -d_k & \text{if } i = e(k) \\ 0 & \text{otherwise} \end{cases}$$

For each vertex $i \in V$ we define the sets $\delta^+(i) = \{(j, j') \in A : j = i\}$ and $\delta^-(i) = \{(j, j') \in A : j' = i\}$. The model uses decision variables x_{ak} that specify the amount of commodity $k \in K$ that flows through arc $a \in A$. Flow is only possible if the arc is “open”. Variable $y_a \in \{0, 1\}$ is one if we choose to open arc a . An open arc $a \in A$ has capacity u_a . There is a cost f_a associated with opening an arc a and a cost c_{ak} for sending one unit of commodity k along arc a . The model for the MCFCND is

$$(M1) \quad \min \sum_{k \in K} \sum_{a \in A} c_{ak} x_{ak} + \sum_{a \in A} f_a y_a \tag{1}$$

subject to

$$\sum_{a \in \delta^+(i)} x_{ak} - \sum_{a \in \delta^-(i)} x_{ak} = d(i, k) \quad \forall i \in V, k \in K \tag{2}$$

$$\sum_{k \in K} x_{ak} \leq u_a y_a \quad \forall a \in A \tag{3}$$

$$x_{ak} \geq 0 \quad \forall a \in A, k \in K \tag{4}$$

$$y_a \in \{0, 1\} \quad \forall a \in A \tag{5}$$

The MCFCND is a versatile model and many vehicle routing and traveling salesman problems can be modelled as a MCFCND with extra constraints. If we for example wish to model the *asymmetric traveling salesman problem* (ATSP) with n cities we add the constraints

$$\sum_{a \in \delta^+(i)} y_a = 1 \quad \forall i \in V$$

$$\sum_{a \in \delta^-(i)} y_a = 1 \quad \forall i \in V$$

and set

$$K = \{1\}, d(i, k) = \begin{cases} n-1 & \text{if } i = 1 \\ -1 & \text{if } i \in V \setminus \{1\} \end{cases}$$

and $c_{a1} = 0$ for all $a \in A$. We set f_a to be the cost of traversing arc $a \in A$ and $u_a = n - 1$. In the *precedence constrained ATSP* (PCATSP) we are given a start node (say node 1) and set of precedence relations $P \subset V \times V$ where each element (i, j) indicates that node i has to precede node j . The ATSP model presented above can be extended to the PCATSP by adding the constraints

$$\sum_{a \in \delta^-(i)} x_{a1} \geq \sum_{a \in \delta^-(j)} x_{a1} \quad \forall (i, j) \in P$$

Consider the capacitated vehicle routing problem (CVRP) where $V = \{0, 1, \dots, n\}$ is the set of nodes, node 0 is the depot and the remaining nodes are customers. We let Q indicate the vehicle capacity, q_i the demand of customer $i \in V \setminus \{0\}$, and we assume that the number of vehicles can be freely chosen. With this we can model the problem as a MCFCND by adding constraints

$$\begin{aligned} \sum_{a \in \delta^+(i)} y_a &= 1 \quad \forall i \in V \setminus \{0\} \\ \sum_{a \in \delta^-(i)} y_a &= 1 \quad \forall i \in V \setminus \{0\} \end{aligned}$$

and we set

$$K = \{1\}, d(i, k) = \begin{cases} \sum_{i=1}^n q_i & \text{if } i = 0 \\ -q_i & \text{if } i \in V \setminus \{0\} \end{cases}$$

and set $c_{a1} = 0$ for all $a \in A$ and we set f_a to be the cost of traversing arc $a \in A$ and $u_a = Q$. In the vehicle routing problem with time windows (VRPTW) where both capacity and time windows constraints are modeled we let $K = \{1, 2\}$ since we both need a demand and a time commodity. For the time commodity we can add upper and lower bounds for each arc in order to model time windows.

Going back to the basic MCFCND model M1 we can obtain an alternative formulation by duplicating each variable, linking each pair of duplicate variables and applying Dantzig Wolfe decomposition with the linking constraints in the master problem. This technique is known as variable splitting (Fisher et al. [1997]) or Lagrangian decomposition (Guignard and Kim [1987]). In the resulting model each variable γ_p represents a “pattern” for a certain vertex $i \in V$. We do not include an i superscript on γ_p (to keep notation simple), but defines a function $v(p)$ that maps a pattern p to its corresponding vertex $i \in V$. A pattern indicates which of the arcs adjacent to vertex i that are open and how much flow they carry. For an arc $a \in \delta^+(v(p))$ and a commodity $k \in K$ the coefficients $x_{akp}^+ \geq 0$ denotes the flow of commodity k on arc a in pattern p and $y_{ap}^+ \in \{0, 1\}$ denotes if arc a is open in pattern p . For all arcs $a \in A \setminus \delta^+(v(p))$ and all $k \in K$ we have $x_{akp}^+ = y_{ap}^+ = 0$. The definition of x_{akp}^- and y_{ap}^- are similar but for arcs $a \in \delta^-(v(p))$. Patterns should be *consistent*, i.e. an arc can only accommodate flow if it has been opened, the flow on each arc should respect the capacity limit and the flow out minus flow in of commodity k should equal $d(i, k)$. Intuitively speaking, a pattern p indicates a feasible set of values of the variables corresponding to arcs adjacent to node $v(p)$. The set of all patterns is denoted Ω and the set of patterns for node i is denoted $\Omega(i)$. With these definitions the model becomes

$$(M2) \quad \min \sum_{k \in K} \sum_{a \in A} c_{ak} \left(\sum_{p \in \Omega} x_{akp}^+ \gamma_p \right) + \sum_{a \in A} f_a \left(\sum_{p \in \Omega} y_{ap}^+ \gamma_p \right) \quad (6)$$

subject to

$$\sum_{p \in \Omega} x_{akp}^+ \gamma_p - \sum_{p \in \Omega} x_{akp}^- \gamma_p = 0 \quad \forall a \in A, k \in K \quad (7)$$

$$\sum_{p \in \Omega} y_{ap}^+ \gamma_p - \sum_{p \in \Omega} y_{ap}^- \gamma_p = 0 \quad \forall a \in A \quad (8)$$

$$\sum_{p \in \Omega(i)} \gamma_p = 1 \quad \forall i \in V \quad (9)$$

$$\sum_{p \in \Omega} y_{ap}^+ \gamma_p \in \{0, 1\} \quad \forall a \in A \quad (10)$$

$$\gamma_p \geq 0 \quad \forall p \in \Omega \quad (11)$$

Here (7) and (8) are the linking constraints that ensures that the selection of patterns are compatible both with respect to x and y variables. Constraint (9) forces a pattern to be selected for each vertex and constraint (10) forces the original y_a variables to be binary.

Model M2 is inspired by a similar model for the Fixed Charge Transportation Problem presented in Mingozzi and Roberti [2017]. A third model, M3, can be obtained by generalizing the *arc quantity* constraints proposed by Mingozzi and Roberti [2017] to the MCFCND. Define $W^+(i, a, k, q) = \{p \in \Omega(i) : x_{akp}^+ = q\}$ to be the set of patterns for vertex i for which arc a is leaving vertex i with q commodities of type k . Similar we define $W^-(i, a, k, q) = \{p \in \Omega(i) : x_{akp}^- = q\}$. With these definitions the arc quantity constraints are

$$\sum_{p \in W^+(i, a, k, q)} \gamma_p - \sum_{p \in W^-(j, a, k, q)} \gamma_p = 0 \quad \forall a = (i, j) \in A, k \in K, q = 0, \dots, u_a \quad (12)$$

We use M3 to denote model M2 plus constraint (12). Let LPM1, LPM2 and LPM3 denote the LP relaxation of model M1, M2 and M3, respectively. For a model X we let $v(X)$ denote its objective value. We have that $v(\text{LPM1}) \leq v(\text{LPM2}) \leq v(\text{LPM3})$. Model LPM2 and LPM3 can grow big and it can be advantageous to use column generation (and row generation) to solve these models.

Using models M2 and M3 we can derive new models and relaxations of the routing problems presented earlier (and many similar routing problems). The LP relaxations can in some cases be strengthened further by including known valid inequalities for the specific routing problems. Furthermore, alternative MCFCND representations for a specific routing problem can lead to different models with better relaxations. All together this results in a framework for developing lower bounds (and exact methods) for many routing problems. Preliminary results indicate that the approach is most promising for TSP problems with side constraints.

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